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$$= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \lambda$$

$$= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)}$$

$$= \lambda e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^{m}}{m!}$$

$$= \lambda e^{-\lambda} e^{\lambda} = \underline{\lambda}.$$

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	Mean	Var	S.D
Bernouli	p	pq	$\sqrt{pq}$
Binomial	np	npq	$\sqrt{npq}$
poission	λ	λ	$\sqrt{\lambda}$

$E \rightarrow$ untill head appear

Que. Calculate the expected value of the

X : No. of ~~head~~ tosses

Sol. ~~$E(X=k)$~~ $E(X) = \sum_{k=0}^{\infty} k \cdot P_X(x=k)$

~~How~~

Ans = 2.

Que. lossing a bag out of 200 in a flight. I will give you RS. 10000/-
Calculate the expected value of the policy (RS=?)

$X = 0$ with prob $1 - \frac{1}{200} = \frac{199}{200}$

$X = -10000/-$ $\rightarrow \frac{1}{200}$

$E(X) = 0 \cdot p(X=0) + (-10000) \cdot \frac{1}{200} = -50/-$

Min. value = 50 + Additional.

Note:- In case of countable infinite case you may not always able to calculate Expected value $E(X)$.

Joint Random Variable:—

let X, Y RV (discreate) in a same sample space

(X, Y)

the joint Random variable will take the value in $\mathbb{R}^2$

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Joint PMF

$= P_{x,y}(x,y) = P(X=x, Y=y) = P[\{(\omega_1, \omega_2), X(\omega_1)=x, Y(\omega_2)=y\}]$

$= P(X=x \text{ and } Y=y)$.

$$\boxed{\sum_x \sum_y P_{X,Y}(x,y) = 1}$$

Ex:- Throwing two ~~coin~~ dice

X = no. of 6

Y = no. of 5

$P(X=0, Y=1) = \frac{\quad}{36}$

not(6)

Que Joint PMF is known to you. Can you find P.M.F for X or Y ??

Yes.

$P_{X,Y}(x,y) \to$ known

$$P_X(x) = \sum_y P_{X,Y}(x,y)$$

$$P_Y(y) = \sum_x P_{X,Y}(x,y)$$

• Marginal P.M.F.

$$E(x,y) = \sum_{y} \sum_{x} ?\; p_{xy}(x,y).$$

As we already discussed in previous classes about conditional prob.,
now we need to define it in term of random variable.

* let $X \rightarrow$ R.V. (discreate)

let $A \subset \Omega \rightarrow$ sample space with $P(A) > 0$

$$P_{X|A}(x) = P(X = x \mid A)$$

$$= P\left(\{w : X(w) = x\} \mid A\right)$$

$$= \frac{P[(X = x) \cap A]}{P(A)} \rightarrow \text{conditional Mass function.}$$

Think
How?

$$\sum_{x} p_{X/A}(x) = ?$$

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$\Omega \rightarrow X \rightarrow E(X) \rightarrow$

$\downarrow$

$P(X) \rightarrow$ P.M.F $\rightarrow$

$\downarrow$

$P(B|A) \rightarrow | \Rightarrow P_{X|A}(x) = P(X=x|A) = \frac{P(X=x \cap A)}{P(A)}$.

$P(A \cap B) = P(A) \cdot P(B)$

Moments of a Random Variable:-

let R.V then k^{th} moment is given by

$$\boxed{E(X^k) = \sum_x x^k P_X(x)}$$

k^{th} Moment around some number say 'a' $\rightarrow$ Proof:-

$$E(X-a)^k = \sum_x (x-a)^k P_X(x).$$

Thm:-

2nd order moment of RV X is minimum when it taken around mean.

$$E[(X - \underset{\Downarrow \; M}{E(X)})^2] \leq E[(X-a)^2]$$

Proof:- $(X-a)^2 = (X - m + m - a)^2$

$= (X-M)^2 + 2(X-M)(M-a) + (M-a)^2$

$$\Rightarrow E((X-a)^2) = E[(X-\mu)^2] + \underbrace{2E[(X-\mu)(\mu-a)]}_{0} + (\mu-a)^2$$

$$\Rightarrow \underline{E[(x-a)^2] \geq E[(X-m)^2]}$$

Thm: 2

for any distribution the first Absolute moment about the mean can not exceed the standard Daviation.

$$E(|X-\mu|) \leq \sigma$$

R.V. $X \to \mu = E(X)$

$\sigma = S.D = \sqrt{Var(X)}$

OR

$$E(|X-\mu|) \leq \sqrt{Var(X)}$$

$$[E(|x-\mu|)]^2 \leq Var(X)$$

Proof:-

$$Var(x) = E[X^2] - (E(x))^2 \geq 0$$

$$\boxed{E(X^2] \geq [E(X)]^2}$$

let $y = |x-\mu|$

$$E[y^2] \geq (E(y))^2$$

OR

$$[E(y)]^2 \leq E[y^2] = E[(|X-\mu|)^2]$$
$$= E[X^2 - 2\mu X + \mu^2]$$
$$= Var(X)$$

$$\Rightarrow [E(y)]^2 \leq var(x)$$

$$\boxed{E(y) \leq \sigma}$$

Ex1- $\Omega = \{1, 2, \dots 6\}$

$A = \{\text{getting even no.}\}$
$= \{2, 4, 6\}$

$$P_{X|A}(k) = \frac{P(X = k \cap A)}{P(A)}$$

let $k = 1$

$$P_{X|A}(1) = \frac{P(X = 1 \cap A)}{P(A)} = \frac{\emptyset}{P(A)}$$

$$P_{X|A}(6) = \frac{1/6}{3/6} = 1/3.$$

- let X & Y - two random variable on the same sample space

& y is known at some particular point y

conditional P.M.F $P_Y(y)$

$$P_{X|Y}(x|y) = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{P_{XY}(x, y)}{P_Y(y)}.$$

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$$\Omega = \biguplus_{i=1}^{n} A_i \quad , \quad B \subset \Omega$$

total prob. $\rightarrow$ $$P(B) = \sum_{i=1}^{n} P(A_i)\, P(B|A_i)$$

$$P(A \cap B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B).$$

$$\begin{cases} P(A) \neq 0 \\ P(B) \neq 0. \end{cases}$$

$$P_{XY}(x,y) = P_Y(y) \cdot P_{X|Y}(x|y)$$
$$= P_X(x)\, P_{Y|X}(y/x).$$

$$E[X|A] = \sum_{x} x\, p_{X|A}(x) \quad \text{— (a)}$$

let $g(x) \rightarrow$ Known

$$E[g(x)|A] = \sum_{x} g(x) \cdot p_{X|A}(x)$$

$$E[X|Y=y] = \sum_{x} x\, p_{X|Y}(x|y).$$

$X \rightarrow$ R.V. $\quad \Omega = \biguplus_{i=1}^{n} A_i$

total Exp.

$$E[X] = \sum_{i=1}^{n} P(A_i)\, E[X|A_i]. \quad \text{— (a)}$$

same

int y.

$$\boxed{\frac{d}{dx} CDF(x) = PDF(x)}$$

(iii) $E[X] = \sum_{i=1}^{n} P(A_i)\, E(X|A)$

(iv) $E[X] = \sum_{y} P_Y(y)\, E[X|Y=y]$.

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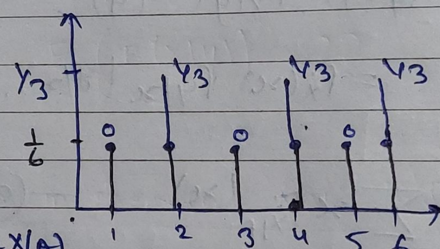
Ex:-

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$A = \{2, 4, 6\}$

$X = i^{th}$

$P(x_i) =$



$P_{(X|A)}(x_i) = P(X = x|A)$

Prob. 1 Tossing a coin 3 times.

X = no. of head $\longrightarrow$ 0, 1, 2, 3

Y = |no. of head − no. of tail| $\longrightarrow$ 1, 3

Calculate (i) PMF for $P(X,Y)$

(ii) Marginal P.M.F of X & Y

(iii) $P(X = x|Y = \;)$.

Ans (i)

$x \backslash y$	1	3
0	0	1/8
1	3/8	0
2	3/8	0
3	0	1/8

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ii) $P_X(x) = \sum p_{X,Y}(x,y)$; $P_Y(y) = \sum p_{X,Y}(x,y)$

$P_X(x) = 0 \cdot 1 + \frac{3}{8} \cancel{+ \frac{3}{8}} + \cancel{3 \cdot (0)} + \cancel{2 \cdot \frac{3}{8}} + \cancel{2 \cdot (0)}$

$+ \cancel{3 \cdot (0)}$

$P_X(x) = 0 + \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + 0 + \frac{3}{8} + 0 + 0 + \frac{1}{8}$

$\uparrow$

Sum of all prob. of x

$=$

iii) $p(X = x \mid Y = 1) = ?$

$= \frac{P(X = x, Y = 1)}{P(Y = 1)} = \frac{P_{X,Y}(x,1)}{P_Y(1)}$

iv) $p(X \mid Y = y) = \frac{p(X = x \mid Y = y)}{} = \frac{P_{X,Y}(x,y)}{P_Y(y) = \sum_x p_{X,Y}(x,y)}$

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$B \subset \Omega = \biguplus A_i$

$P(B) = P(A_1)\,P(B/A_1) + P(A_2)\cdot P(B/A_2) + \cdots$

$\sum x\, p_x(y) = x\,P(A_1)\,P(x=x\,|\,A_1) + x\,P(A_2)\,P(x=x\,|\,A_2) + \cdots$

$E(X) = P(A_1)\,E(X/A_1) + + \cdots + P(A_n)\,E(X/A_n).$

* $E(x|y] = \sum_x x\, p_{xy}(x|y)$

Homework?? $E(E(X|Y)) = ??$

- let two independent R.V. x & y

$p_{xy}(x|y) = p_x(x)\,p_y(y) \;;\; \forall\; x, y$

Ex:- (X, y)

$X = 0, 1, 2, 3$

$Y = 1, 2, 3, 4$

P.M.F $\quad p_{x,y}(x,y) = \dfrac{3x+4y}{232}$

$$\left[\sum_x \sum_y p_{x,y}(x,y) = 1\right]$$

Calculate (i) $P(X \geq 2,\; y \leq 3)$

$P(x \geq 2, y \leq 3) = P(x=2, y = 1, 2, 3) + P(x=3, y = 1, 2, 3)$

i.e. $P(x=2, y=1) + P(x=2, y=2) + \ldots$

$$= \sum_{x=2}^{3} \sum_{y=1}^{3} p_{x,y}(x, y)$$

(ii) x & y are independent = ?

$p_{x,y}(x,y) = p_x(x) \cdot p_y(y)$ ← condition.

$\Rightarrow \boxed{p_{x,y}(1,2) \neq p_x(1)\, p_y(2)}$

* let given that x & y are independent

(i) $E[X \cdot Y] = ? = E(X) \cdot E(Y)$

(ii) $Var[X+Y] = ? = Var(X) + Var(Y)$.

Proof:-

(i) $E(X \cdot Y) = \sum_x \sum_y xy \, p_{x,y}(x,y)$

$$= \sum_x \sum_y xy \, p_x(x) \cdot p_y(y)$$

$$= \sum_x x \, p_x(x) \cdot \sum_y y \, p_y(y)$$

$\boxed{E(xy) = E(X) \cdot E(Y)}$

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(ii) $Var(x+y) = Var(x) + Var(y)$

$\bar{x} = x - E(x)$
$\bar{y} = x - E(y)$

$V(\bar{x}+\bar{y}) = Var(\bar{x}) + Var(\bar{y})$

$\Downarrow$

$V(x+y) = Var(x) + Var(y)$

$$\left\{ \begin{array}{l} Var(x) \\ = (E(x^2) - E(x))^2 \end{array} \right.$$

$V(\bar{x}+\bar{y}) = E[(\bar{x}+\bar{y})^2]$

$= E(x^2 + 2\bar{x}\bar{y} + \bar{y}^2)$

$V(\bar{x}+\bar{y}) = E(\bar{x})^2 + \cancel{E(\bar{x})E(\bar{y})} + E(\bar{y})^2$

0

$$\boxed{V(\bar{x}+\bar{y}) = V(\bar{x}) + V(\bar{y})}$$

NOTE:- PMF of the discreate random variable that is uniformly distibuted b/w two integer a and b. then its mean and variance are

Mean $= E(x) = \frac{a+b}{2}$; $Var(x) = \frac{(b-a)(b-a+2)}{12}$

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$X_1, X_2, \dots, X_n \sim$ Bernoulli random Variable

$X = X_1 + X_2 + \dots + X_n$.

$X \rightarrow$ Binomial

$E(X) = E(X_1) + E(X_2) + \dots + E(X_n)$

$1.P + \dots + 1.P$

$E(X) = nP$

$Var(X) = Var(X_1) + \dots + Var(X_n)$

$= np(1-P)$

* $X \sim B(n_1, P_1)$; $y \sim B(n_2, P_2)$

$X + y \sim B(?, ?)$

* $X \sim Pois(\lambda_1)$; $y \sim pois(\lambda_2)$

$X + y \sim pois(?)$

* $X \sim B(n, P)$

$y \sim pois(\lambda)$

Can I add $X + y = ?$

if yes then $X+y \rightarrow$ type of random Variable

* $P(a \leq x \leq b) = \int_a^b f_X(x).dx$

$f_X(x)$ is called density function.

i) $f_X(x) \geq 0 \ \forall x$

ii) $\int_{-\infty}^{\infty} f_X(x) \cdot dx = 1$

Que. $f_X(x) = \begin{cases} Kx(1-x) & ; \ 0 \leq x < 1 \\ 0 & ; \ o.w \end{cases}$

$P(a \leq X \leq b) = P(a < x \leq b) = P(a \leq X < b)$

Uniform Random Variable:—

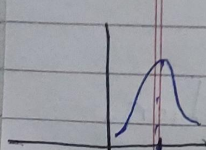
$X \sim \text{Union}(a, b)$

PDF: $f_X(X) = \begin{cases} \frac{1}{b-a} & ; \ a < X < b \\ 0 & ; \ o.w \end{cases}$

$E(X) = \int_a^b x f_X(x) dx = \frac{1}{b-a} \int_a^b x dx = \frac{a+b}{2}$

$Var(X) = E(X^2) - (E(X))^2 = \frac{(b-a)^2}{12}$

Gaussian RV / Normal R.V:—



$X \sim N(U, \sigma^2)$

$x = \mu$

$f_X(X) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \ ; \ -\infty < x < \infty$

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$f_X(x) \geq 0 \quad \forall x$

$E(X) = \mu$

$Var(X) = \sigma^2$

Exponential RV:—

$X \sim Exp(\lambda)$

PDF $\quad f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & o.w \end{cases}$

$E(X) = 1/\lambda$

$Var(X) = \frac{1}{\lambda^2}$

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let X be a random variable is said to ~~to~~ continuous if there exist $f_X(x)$ defined for all $X \in (-\infty, \infty)$

Such that:-

$$P(X \in B) = \int_B f_X(x) \cdot dx$$

$f_X(x) \rightarrow$ Called Prob. density func" (PDF).

① $f_X(x) \geq 0$

$B = [a, b]$ then $\quad P(a \leq x \leq b) = \int_a^b f_X(x) \cdot dx.$

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(ii) $P(x=a) = \int_a^a f_x(x)\, dx = 0$

(i) $P(a<X\le b) = P(a\le X\le b) = P(a<x<b)$

(ii) $P(-\infty < x < \infty) = \int_{-\infty}^{\infty} f_x(x)\, dx = 1.$

Example:- Picking a random no. from the interval $[0,1]$.

Example:- life time of a machine.

Uniform Random Variable:-

$X \rightarrow$ R.V. is called uniformly on the interval (a,b) if prob. density func.

$$F_x(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & o.w \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x\, f_x(x)\, dx = \frac{a+b}{2}$$

$$Var(X) = E(X^2) - ((E(x))^2) = \frac{(b-a)^2}{12}$$

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Ex. 10 10:15 10:30 uniform from (10 to 10:30).

P(waiting less than 5 min).

$\uparrow$ min

$X \sim$ union $(0 \to 30)$.

$$f_X(x) = \begin{cases} \frac{1}{30} & 0 < x < 30 \\ 0 & \text{o.w} \end{cases}$$

$$P(10 < x < 15) + P(25 < x < 30)$$

$$= \int_{10}^{15} \frac{1}{30} dx + \int_{25}^{30} \frac{1}{30} dx$$

$$= \frac{1}{30} \times 5 + \frac{1}{30} \times 5 \Rightarrow \frac{1}{3}.$$

Guassian RV / Normal RV :-

X is said to be Normal R.V. if ither the prob. density fun"

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

$$* \quad = \int_{-\infty}^{\infty} f_X(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

$$\boxed{\frac{x-\mu}{\sigma} = z}$$

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$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} . dz$$

let $I = \cancel{\frac{1}{\sqrt{2\pi}}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} . dz$

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} . dz\right)\left(\int_{-\infty}^{\infty} e^{-y^2/2} . dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(z^2+y^2)/2} \, dz \, dy$$

$\left(\begin{array}{l} z = r\cos\theta \\ y = r\sin\theta \end{array} \right.$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-r^2/2} . r dr d\theta$$

$$= 2\pi \int_0^{\infty} e^{-r^2/2} r dr = \int_0^{\infty} e^{-R} . dR = 2\pi$$

$$\Rightarrow \boxed{I = \sqrt{2\pi}}$$

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$$E(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \cdot e^{-\frac{(x-\mu)^2}{\sigma^2}} \cdot dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + \mu)\, e^{-z^2/2} \, .dz$$

$$= \frac{1}{\sqrt{2\pi}} \left[\sigma \int_{-\infty}^{\infty} z\, e^{-z^2/2} \, .dz + \mu \int_{-\infty}^{\infty} e^{-z^2/2} \, .dz \right]$$

$\Downarrow$

μ

E(X)

$$\boxed{E(x). = \frac{\mu}{\sqrt{2\pi}}} \; ?$$

$$Var(x) = E(x^2) - (E(x))^2$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \; e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad - \frac{\mu^2}{\cancel{2\pi}}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + \mu)^2 \, e^{-z^2/2} \, .dz \Big) - \mu^2$$

$$= \sigma^2$$

<u>Integration by parts:</u>—

$$\int f(x) g(x) . dx = f(x) \int g(x) - \int \left(\frac{d}{dx} (f(x) \cdot \int g(x) dx \right)$$

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$X \sim N(\mu, \sigma^2)$

$Z : \frac{X-\mu}{\sigma} \sim N(0,1)$

Standard Normal R.V.

$E(Z) = \frac{E(X)-\mu}{\sigma} = \frac{\mu-\mu}{\sigma} = 0$

$Var(Z) = \frac{1}{\sigma^2} Var(X) = \frac{\sigma^2}{\sigma^2} = 1,$

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X be the R.V. said to Exponential if

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & ; x \geq 0 \ \& \ \boxed{\lambda > 0} \\ 0 & \text{o.w} \end{cases}$$

$E(X) = \lambda$, $Var(X) = \frac{1}{\lambda^2}$

CDF or Distributive function

$= P(x \leq x) = P(-\infty < x \leq x)$

$= P(\{\omega : -\infty < X(\omega) \leq x\})$

CDF is ↑ function or non decreasing (monotonically).

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let $x < y$ $-\infty$ —— x —— y —— ∞

$(-\infty < x \leq y) = (-\infty < X < x)$
$+ P(x < X \leq y)$

$F_X(y) - F_X(X) = P(X < x < y) \geq 0$

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Example:—

$f_X(x) = \begin{cases} C_1 & ; 15 < x < 20 \quad P(\text{*}) = 2/3 \\ C_2 & ; 20 < x < 25 \quad P(\ldots) = 1/3 \\ 0 & \text{o.w} \end{cases}$

$P(\text{*}) = \frac{2}{3} = P(15 < x < 20) = \int_{15}^{20} f_X(x)dx$

$= C_1(5).$

$\Rightarrow C_1(5) = 2/3$

$\boxed{C_1 = \frac{2}{15}}$

Similarly

$\boxed{C_2 = \frac{1}{15}}$

CDF $F_X(x) = P(X \leq x).$

$F_X(x) \to 0$ on $x \to 0$

$\to 1$ on $x \to \infty$

① $X \rightarrow$ Discreate then $F_x \rightarrow$ step function.

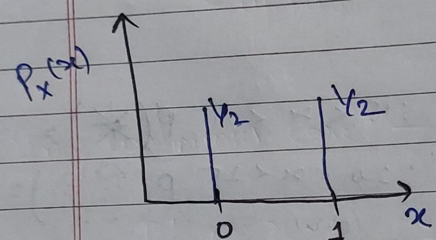
if $X \rightarrow$ Continuous then $f_x \rightarrow$ continuous.

Example:—

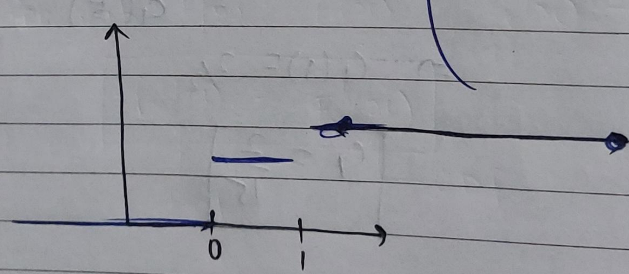
A coin toss

$X = 1$ {Head} $\Rightarrow P(X=1) = 1/2$

$X = 0$ {tail} $\Rightarrow P(X=0) = 1/2$



$$F_X(x) = P(X \le x) = \begin{cases} 0 & x < 0 \\ 1/2 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$



Example:—

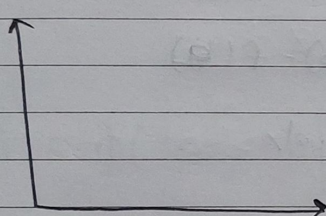
$$f_X(x) = \begin{cases} 6x(1-x) & ; \ 0 < x < 1 \\ 0 & ; \ o.w \end{cases}$$

Calculate CDF $F_X(x)$.

$$F_X(x) = P(X \le x) = \begin{cases} 0 \; ; & x < 0 \\ x^2[3-2x] \; ; & 0 \le x < 1 \\ 1 \; ; & x \ge 1 \end{cases}$$

$$\int_0^x 6(x(1-x)) \cdot dx = 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^x$$

$$\Rightarrow x^2(3-2x)$$



continuous graph $\Rightarrow$ continuous function.

(*) Normal Random Variable :-

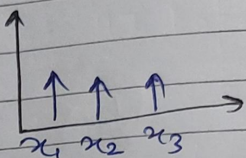
$$X \sim N(\mu, \sigma^2)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \; ; \quad -\infty < x < \infty$$

$$Z = \frac{X-\mu}{\sigma} \sim N(0, 1).$$

let $F_X(x)$ is given:-

In discreate Case:-



E $P(X=3) = ?$

$\Rightarrow F_X(x_3) - F_X(x_2)$

$\uparrow$ up to 3 $\quad$ $\uparrow$ up to 2

In Continuous Case:-

$$P(a \le x \le b) = F(b) - F(a)$$

$$= \int_a^b f_X(x) \cdot dx$$

Relationship b/w $F(x)$ & $f(x)$:-

$$\boxed{f_X(x) = F_X'(x)}$$

PDF $\quad$ CDF

* CDF $\quad X \sim N(\mu, \sigma^2)$

$$F_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}} \cdot dt$$

$Z \sim N(0,1)$

$$F_Z(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} \, dt$$

$\Downarrow$

$\Phi(z)$

↳ we always use this notation for CDF for standard Normal.

$X \sim N(\mu, \sigma^2)$

$F_X(x) = P(X \le x).$

$$Z = \frac{X-\mu}{\sigma} \quad = P(\sigma Z + \mu \le x)$$

$$= P\left(Z \le \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

$$\boxed{F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)}$$

Remarks:-

$$\boxed{\Phi(-z) = 1 - \Phi(z)} \quad \forall z.$$

Proof:-

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} . dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2} . dt - \frac{1}{\sqrt{2\pi}} \int_{z}^{+\infty} e^{-t^2/2} . dt$$

$$= 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-z} e^{-t^2/2} . dt$$

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$\Phi(z) = 1 - \Phi(-z)$

Example:—

Annual Snawfall at IIT Ropar $\sim N(\underset{\text{mean}}{60}, \underset{\text{var.}}{400})$

accumulation of snow in term of inches $P(X \geq 80)$

$X \sim N(60, (20)^2)$

$\mu = 60, \sigma = 20$

$$Z = \frac{X-\mu}{\sigma} = \frac{X-60}{20}$$

$$\Rightarrow \boxed{X = 20Z + 60}$$

$$P(X \geq 80) = P((20Z+60) > 80)$$

$$= P(Z \geq 1) = 1 - P(Z \leq 1)$$

$$= 1 - \Phi(1)$$

$$\Phi(1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{1} e^{-t^2/2} \cdot dt$$

Dataset for $\Phi(z)$ will ~~g~~ be given

$X \to E(X)$

$Y = g(X)$. $\quad Y \to f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$

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Continuous Random Variable: —

$X \rightarrow$ Unif | Normal | Exp.

PDF $\rightarrow f_X(x)$ $\qquad F_X'(x) = f_X(x)$

CDF $\rightarrow F_X(x)$

$X \sim N(\mu, \sigma^2)$, $\quad Z = \dfrac{X-\mu}{\sigma} \sim N(0,1)$.

- $P(a < x \leq b) = F_X(b) - F_X(a)$
- $X \sim N(\mu, \sigma^2)$, $\quad Z = \dfrac{x-\mu}{\sigma}$
- $F_X(x) = \Phi\left(\dfrac{x-\mu}{\sigma}\right)$, $\quad \Phi(z) = 1 - \Phi(-z)$.

Example: —

let, weight of students in the class follow $N(\mu, \sigma^2)$, $\mu = 40$kg, $\sigma = 5$kg

find % of students that have

i) weight greater than 40

ii) b/w 38 kg to 52 kg.

Sol.: (i) $P(X > 40) = 1 - P(X \leq 40)$

$= 1 - P\left(\dfrac{X-\mu}{\sigma} \leq \dfrac{40-\mu}{\sigma}\right)$

$= 1 - P\left(Z \leq \dfrac{40-40}{5}\right)$.

$$= 1 - P(Z \le 0)$$

$$= 1 - \Phi(0) = 1 - \frac{1}{2} = \frac{1}{2}.$$

ii) $$P(38 < x \le 52) = \int_{38}^{52} f_X(x)\cdot dx$$

$$= P\left(\frac{38-\mu}{\sigma} < \frac{x-\mu}{\sigma} < \frac{52-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{52-\mu}{\sigma}\right) - \Phi\left(\frac{38-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{52-40}{5}\right) - \Phi\left(\frac{38-40}{5}\right)$$

$$X \rightarrow \boxed{Y \rightarrow g(x)}$$
↓
PDF

- funⁿ monotonic increasing

$X \searrow x$

$Y \searrow g(y) \Rightarrow y = g(x).$

$$F_X(x) = P(X \le x) = P(g^{-1}(y) \le g^{-1}(y))$$

Diffⁿ

$$\boxed{f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|}$$

Ex:-
find

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Ex:- $X \sim N(0,1)$.

find CDF for $Y = e^X$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad -\infty < x < \infty$$

$$\frac{dy}{dx} = e^x$$

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-(\log y)^2/2} \cdot \frac{1}{y}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{y} e^{-\frac{(\log y)^2}{2}} \quad 0 < y < \infty$$

$$F_Y(y) = P(Y \le y) = \int_{-\infty}^{y} f_Y(z) \cdot dz$$

f(x). y, ∞

* $X \sim$ R.V. pdf

$$f_X(x) = \frac{1}{2} e^{-|x|} \quad -\infty < x < \infty$$

CDF for $Y = X^2$

⊗ $X \sim U \cdot N(1, 50)$

$$P\left(X + \frac{125}{X} > 40\right) = ??$$

$$\int_{x_1}^{x_2} P(? < x < ?)$$

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$$P(x^2 + 125 > 40x) = P(|X-20| > 5\sqrt{11}).$$

$\Leftrightarrow x^2 + 125 > 40X$

$x^2 - 40X > -125$

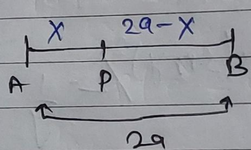
$X^2 - 40X + (20)^2 > -125 + 400$

$(X-20)^2 > (5\sqrt{11})^2$

$|X-20| > 5\sqrt{11}$

Example:–

H.W.



$P(\text{(area of Rec with side } AP \,\&\, PB) > \frac{a^2}{2}) = ??$

$\Rightarrow P\left(X(2a-X) > \frac{a^2}{2}\right) = ??$

⊛ $X, Y \rightarrow$ cts R.V.

$(X,Y) \in B \subseteq \mathbb{R}^2$

Joint Prob. D.F.

$$P((x,y) \in B) = \iint_{(x,y)\in B} f_{X,Y}(x,y)\,.dy.dx.$$

$f_{X,Y}(x,y) \geq 0$

Joint PDF

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Marginal PDF:—

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx.$$

if X, Y are independent then $f_{X,Y}(x,y) = f_X(x) f_Y(y)$

Joint CDF:— X, Y independent $F_{X,Y}(x,y) = F_X(x) \cdot F_Y(y)$.

$$F_{X,Y}(x,y) = P(X \leq x,\ Y \leq y)$$

$$= \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(x,y)\, dy\, dx.$$

Marginal:—

$$F_X(x) = \int_{-\infty}^{(\infty)} F_{X,Y}(x,y) \cdot dy$$

Joint PDF

Jacobian,

$$f_{U,V}(u,v) = f_{X,Y}(x,y) \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$$

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

21/02/24 (Extra class). Question Practice class.

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(1) $X \sim RV$

$$f_X(x) = \begin{cases} 1 & ; 0 \le x \le 1 \\ 0 & ; \text{o.w} \end{cases}$$

find PDF for e^x. ?

Gen. Method:—

let $Y = e^X$

CDF → Y,

$$F_Y(y) = P(Y \le y)$$

$$= P(e^X \le y) = P(X < \log y)$$

$$= \int_{-\infty}^{\log y} f_Y(x)\,dx = \int_0^{\log y} 1 \cdot dx = \log y$$

⇒

$$F_Y(y) = \log y$$

$$\Rightarrow f_Y(y) = \begin{cases} \frac{1}{y} & 1 \le y \le e \\ 0 & \text{o.w.} \end{cases}$$

(2) (X, Y) → given

$U = X+Y$, $V = \frac{X}{X+Y}$ ⇒ $x = UV$

& $y = U - VU$

$U(1-V)$.

Calculate Joint PDF for $(u,v) = ?$

let $X \to x$, $Y \to y$
$U \to u$, $V \to v$

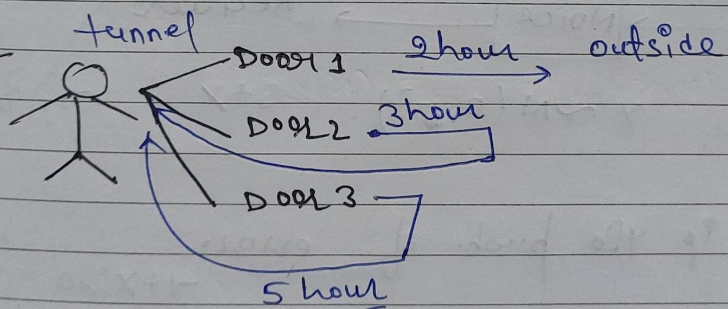
In real variable $x = uv$, $y = u(1-v)$.

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 1-v & -u \end{vmatrix} = -u$$

So

$$f_{U,V}(u,v) = f_{X,Y}(x,y) \cdot u$$

Q3



Calculate Expected time that this men come out safely.

Ans let $X \to$ expected time.

$$E[X] = E[X \mid \text{Door1}]\,P(\text{Door1}) + E[X \mid \text{Door2}] \cdot P(\text{Door2}) + E[X \mid \text{Door3}]\,P(\text{Door3}).$$

$$= \frac{1}{3}\left(E[X|\text{Door}1] + E[X|\text{Door}2] + E[X|\text{Door}3]\right)$$

$$E(X) = \frac{1}{3}\left[2 + 3 + E(X) + 5 + E(X)\right]$$

$$\Rightarrow \quad E(X) = \frac{1}{3}\left[10 + 2E(X)\right]$$

$$3E(X) - 2E(X) = 10$$

$$\boxed{E(X) = 10}$$

(4) Signal Dectation problem :—

$$\text{Transmitter} \longrightarrow \boxed{\text{Noise}} \longrightarrow \text{Reciver} \begin{cases} +1, & \text{if } X+S \geq 0 \\ -1, & \text{if } X+S < 0 \end{cases}$$

$S = +1$, $= -1$ $\quad X \sim N(0, \sigma^2)$ $\quad S+X$

What is the prob. of error.

Sol: Error occure when -1 is transmited $S+X \geq 0$ $\quad -1+X \geq 0 = \boxed{X \geq 1}$

or $+1$ is transmited $S+X < 0$

$$\Rightarrow \quad 1+X < 0$$

$$\boxed{X < -1}$$

$$f_y(y) = \sum_{x} P_{x,y}(x,y).$$

(5) $(X,Y) \rightarrow$ R.V.s $\rightarrow$ Discreate

Joint PMF $P_{X,Y}(1,1) = 0.5$; $P_{X,Y}(1,2) = 0.1$

$P_{X,Y}(2,1) = 0.1$, $P_{X,Y}(2,2) = 0.3$

$(X,Y) \rightarrow (1,1), (1,2), (2,1), (2,2)$

conditional

PMF of x given that $y=1$.

Sol:

$$P_{X|Y}(X|Y=1) = \frac{P(X=x, Y=1)}{P(Y=1)} = \frac{P(X,1)}{0.6}$$

$$P(Y=1) = P_{X,Y}(1,1) + P(2,1) = 0.5 + 0.1$$
$$= 0.6$$

So

$$P_{X|Y}(X|Y=1) = \frac{P(X,1)}{0.6}$$

$$P_{1|1}(X=1|Y=1) = \frac{P(1,1)}{0.6} = \frac{5}{6}$$

$$P_{2|1}(X=2|Y=1) = \frac{0.1}{0.6} = \frac{1}{6}.$$

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Date $0 < x, y < 1$

(6) X, Y given $f_{X,Y}(x,y) = \begin{cases} 6xy(2-x-y) & \\ 0 & o.w \end{cases}$

Calculate Conditional $E[X \mid Y = y] = ??$

Recall

$E[X \mid Y = y]$

$= \int_{-\infty}^{\infty} x \, f_{X \mid Y}(x \mid y) \cdot dx$

& $f_{X \mid Y}(x \mid y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$